

# CONCENTRATION PHENOMENA FOR FRACTIONAL ELLIPTIC EQUATIONS INVOLVING EXPONENTIAL CRITICAL GROWTH

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ABSTRACT. In this paper, we deal with the following singular perturbed fractional elliptic problem  $\varepsilon(-\Delta)^{1/2}u + V(z)u = f(u)$  in  $\mathbb{R}$ , where  $(-\Delta)^{1/2}u$  is the square root of the Laplacian and  $f(s)$  has exponential critical growth. Under suitable conditions on  $f(s)$ , we construct a localized bound state solution concentrating at an isolated component of the positive local minimum points of the potential of  $V$  as  $\varepsilon$  goes to 0.

## CONTENTS

1. Introduction	1
1.1. Statement of the main the result	3
1.2. Outline	3
2. Preliminary results	4
3. The Caffarelli and Silvestre's method	6
4. Proof of Theorem 1.1	19
References	19

## 1. INTRODUCTION

In this paper, we are concerned with existence and concentration of positive solutions for the following singular perturbed fractional elliptic problem

$$(P_\varepsilon) \quad \begin{cases} \varepsilon(-\Delta)^{1/2}u + V(z)u = f(u) & \text{in } \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}), \quad u > 0 & \text{on } \mathbb{R}, \end{cases}$$

where  $\varepsilon$  is a small positive parameter, the potential  $V$  is bounded away from zero, the nonlinearity  $f(s)$  has exponential critical growth and  $(-\Delta)^{1/2}u$  is the square root of the Laplacian, which may be defined for smooth functions as

$$\mathcal{F}((-\Delta)^{1/2}u)(\xi) = |\xi|\mathcal{F}(u)(\xi),$$

where  $\mathcal{F}$  is the Fourier transform, that is,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi \cdot x} \phi(x) \, dx,$$

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for functions  $\phi$  in the Schwartz class. Also, for sufficiently smooth  $u$ ,  $(-\Delta)^{1/2}u$  can be equivalently represented, see [17, 22], as

$$(-\Delta)^{1/2}u = -\frac{1}{2\pi} \int_{\mathbb{R}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2} dy,$$

and, by [17, Propostion 3.6],

$$\|(-\Delta)^{1/4}u\|_{L^2}^2 := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^2} dx dy, \quad \forall u \in H^{1/2}(\mathbb{R}).$$

Here  $H^{1/2}(\mathbb{R})$  is the fractional Sobolev space

$$H^{1/2}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \|(-\Delta)^{1/4}u\|_{L^2}^2 < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^{1/2}} = \left( \|u\|_{L^2}^2 + \|(-\Delta)^{1/4}u\|_{L^2}^2 \right)^{1/2}.$$

We suppose that the potential  $V : \mathbb{R} \rightarrow \mathbb{R}$  is bounded and satisfies the following hypotheses:

(V<sub>1</sub>):  $V$  is locally Hölder continuous and there exists  $V_0 > 0$  such that

$$V(z) \geq V_0, \quad \forall z \in \mathbb{R},$$

(V<sub>2</sub>): there exists a bounded interval  $\Lambda \subset \mathbb{R}$  such that

$$V_0 \equiv \inf_{\Lambda} V(z) < \min_{\partial\Lambda} V(z).$$

The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies the so-called Ambrosetti-Rabinowitz condition, introduced in [5], namely,

(AR) there exists  $\vartheta > 2$  with  $0 < \vartheta F(s) \leq s f(s)$  for all  $s > 0$ ,  $F(s) = \int_0^s f(t) dt$ .

In addition to the above condition we make the following assumptions on  $f$ :

(f1) :  $f : \mathbb{R} \rightarrow \mathbb{R}^+$  is  $C^1$  function with  $f(s) = 0$  if  $s < 0$ .

(f2) :  $f(s) = o(s)$  near origin.

(f3) :  $f(s)/s$  is an increasing function in  $\mathbb{R}^+$ .

(f4) : There exist constants  $p > 2$  and  $C_p > 0$  such that

$$f(s) \geq C_p s^{p-1} \quad \text{for all } s > 0,$$

where

$$C_p > \left[ \beta_p \left( \frac{2\vartheta}{\vartheta - 2} \right) \frac{1}{\min\{1, V_0\}} \right]^{(p-2)/2},$$

with

$$\beta_p = \inf_{\mathcal{N}_0} \tilde{J}_0,$$

$$\mathcal{N}_0 = \{v \in X^1(\mathbb{R}_+^2) \setminus \{0\} : \tilde{J}_0'(v)v = 0\}$$

and

$$\tilde{J}_0(v) = \frac{1}{2} \int_{\mathbb{R}_+^2} |\nabla v|^2 dx dy + \frac{1}{2} \int_{\mathbb{R}} V_0 |v(x, 0)|^2 dx - \frac{1}{p} \int_{\mathbb{R}} |v(x, 0)|^p dx.$$

where  $X^1(\mathbb{R}_+^2)$  is defined in (2.1).

We are interested in bound state solution of  $(P_\varepsilon)$  (solution with finite energy), when  $f$  has the maximal growth which allow us to treat problem  $(P_\varepsilon)$  variationally in the fractional Sobolev space  $H^{1/2}(\mathbb{R})$  motivated by the following Trudinger-Moser type inequality due to T. Ozawa [24].

**Theorem A.** *There exists  $0 < \omega \leq \pi$  such that, for all  $\alpha \in (0, \omega)$ , there exists  $H_\alpha > 0$  with*

$$(1.1) \quad \int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx \leq H_\alpha \|u\|_{L^2}^2,$$

for all  $u \in H^{1/2}(\mathbb{R})$  with  $\|(-\Delta)^{1/4} u\|_{L^2}^2 \leq 1$ .

In view of (1.1), we say that  $f$  has exponential critical growth at  $+\infty$ , if there exist  $\omega \in (0, \pi)$  and  $\alpha_0 \in (0, \omega)$ , such that

$$\lim_{s \rightarrow +\infty} \frac{f(s)}{e^{\alpha s^2}} = 0, \quad \forall \alpha > \alpha_0, \quad \text{and} \quad \lim_{s \rightarrow +\infty} \frac{f(s)}{e^{\alpha s^2}} = +\infty, \quad \forall \alpha < \alpha_0.$$

**1.1. Statement of the main the result.** The following theorem contains our main result:

**Theorem 1.1.** *Assume  $(V_1), (V_2), (AR)$ , and  $(f1) - (f4)$  hold. Then there exists  $\varepsilon_0 > 0$  such that for  $\varepsilon \in (0, \varepsilon_0)$ , problem  $(P_\varepsilon)$  possesses a positive bound state solution  $u_\varepsilon(z)$  verifying the following conditions*

- I):**  $u_\varepsilon$  has at most one local (hence global) maximum  $z_\varepsilon$  in  $\mathbb{R}$  and  $z_\varepsilon \in I$ ,
- II):**  $\lim_{\varepsilon \rightarrow 0^+} V(z_\varepsilon) = V_1 = \inf_I V$ ,

This result extends to the nonlocal case the main result in [19]. The proof is made combining Ozawa inequality [24] with Del Pino and Felmer [16] truncation argument and a recent approach developed in Alves and Miyagaki [4]. In [12, 25, 27] were established existence results in nonlocal situation, while in [13, 14, 20, 26] a concentration phenomena were proved imposing a global condition in  $V$ .

**Remark 1.2.** (1) We recall that the condition  $(AR)$  impose some superquadratic growth condition on the nonlinearity  $F$ .

- (2) The condition  $(f4)$  appeared first in [11], then for instance in [2] and [19]. For the non-local situation it was used, e.g., in [18].
- (3) Critical growth of Trudinger-Moser type was used in [15], also in [1, 2, 19]. In [23] and in [18] were used the Ozawa inequality to discuss nonlocal problem in bounded and unbounded domain, respectively.
- (4) Notice that, if  $f(s)$  has exponential critical growth, instead of assumption  $(f4)$ , it is enough to assume that there exist  $p > 2$  and  $\mu > 0$  such that

$$\liminf_{s \rightarrow 0^+} \frac{f(s)}{s^{p-1}} \geq \mu.$$

Throughout the paper, unless explicitly stated, the symbol  $C$  will always denote a generic positive constant, which may vary from line to line.

**1.2. Outline.** The sequel of the paper is organized as follows. The next section contains some technical results, which are crucial tools to prove our main theorem. In Sect. 3, we adapt a method due to L. Caffarelli and L. Silvestre to obtain a local realization of the fractional Laplacian via a Dirichlet-to-Neumann operator. As a consequence of this argument we transform our nonlocal Problem  $(P_\varepsilon)$  into one local problem defined on the upper half plane  $(LP_\varepsilon)$ . Using variational techniques combined with Del Pino and Felmer truncation argument we give the proof of Theorem 1.1 in Sect. 4,

## 2. PRELIMINARY RESULTS

In this section we collect preliminary facts for future reference. First of all, let us set the standard notations to be used in the paper. We denote the upper half-space in  $\mathbb{R}^2$  by  $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ . In the sequel,  $X^1(\mathbb{R}_+^2)$  denotes the completion of  $C_0^\infty(\overline{\mathbb{R}_+^2})$  with relation to the norm  $\|v\|_\varepsilon$ ,

$$(2.1) \quad \begin{aligned} X^1(\mathbb{R}_+^2) &:= \overline{C_0^\infty(\mathbb{R}_+^2)}^{\|\cdot\|_\varepsilon}, \quad \text{where} \\ \|v\|_\varepsilon &:= \left( \int_{\mathbb{R}_+^2} |\nabla v(x, y)|^2 dx dy + \int_{\mathbb{R}} V(\varepsilon x) |v(x, 0)|^2 dx \right)^{1/2}. \end{aligned}$$

Moreover, we denote by  $\|\cdot\|$  the usual norm in  $X^1(\mathbb{R}_+^2)$ , that is,

$$\|v\| = \left( \int_{\mathbb{R}_+^2} |\nabla v(x, y)|^2 dx dy + \int_{\mathbb{R}} |v(x, 0)|^2 dx \right)^{1/2}.$$

Since the potential  $V$  is bounded from above and below, it is easy to see that  $\|\cdot\|_\varepsilon$  and  $\|\cdot\|$  are equivalent norms in  $X^1(\mathbb{R}_+^2)$  with

$$(2.2) \quad \min\{1, V_0\} \|v\| \leq \|v\|_\varepsilon \leq \min\{1, |V|_\infty\} \|v\|, \quad \forall v \in X^1(\mathbb{R}_+^2).$$

Using the above definition, we see that if  $v \in X^1(\mathbb{R}_+^2)$ , then  $u(x) = v(x, 0)$  belongs to  $H^{1/2}(\mathbb{R})$  and

$$\|v\| = \|u\|_{H^{1/2}}.$$

Since  $H^{1/2}(\mathbb{R})$  is continuously embedded into  $L^q(\mathbb{R})$  for all  $q \geq 2$ , c.f. [17, Theorem 6.9], it follows that  $X^1(\mathbb{R}_+^2)$  is also continuously embedded into  $L^q(\mathbb{R})$  for all  $q \geq 2$ . Moreover, the embedded

$$X^1(\mathbb{R}_+^2) \hookrightarrow L^q(A)$$

are compact for any bounded measurable set  $A \subset \mathbb{R}$ . See [22, Proposition 3.6] also [18, Remark 2.1].

Our first lemma is an important Trudinger-Moser inequality on  $X^1(\mathbb{R}_+^2)$ , which was proved in [18, Lemma 2.4].

**Lemma 2.1.** *Let  $(v_n) \subset X^1(\mathbb{R}_+^2)$  be a bounded sequence and assume  $\sup_{n \in \mathbb{N}} \|v_n\|^2 = M$ . Then*

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}} (e^{\alpha |v_n(x, 0)|^2} - 1) dx < \infty, \quad \text{for every } 0 < \alpha < \frac{\omega}{M^2};$$

*In particular, if  $M \in (0, 1)$ , there exists  $\alpha_M < \omega$  such that*

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}} (e^{\alpha_M |v_n(x, 0)|^2} - 1) dx < \infty.$$

Using the above lemma, we are able to prove some technical lemmas. The first of them is crucial in the study the behavior of Palais-Smale sequences.

**Lemma 2.2.** *Let  $(v_n)$  be a sequence in  $X^1(\mathbb{R}_+^2)$  with*

$$(2.3) \quad \limsup_{n \rightarrow +\infty} \|v_n\|^2 < 1.$$

*Then, there exist  $t > 1$  sufficiently close to 1 and  $C > 0$  satisfying*

$$\int_{\mathbb{R}} \left( e^{\omega |v_n(x, 0)|^2} - 1 \right)^t dx \leq C, \quad \forall n \in \mathbb{N}.$$

**Proof.** Using (2.3) there are  $m > 0$  and  $n_0 \in \mathbb{N}$  verifying

$$\|v_n\|^2 < m < 1, \quad \forall n \geq n_0.$$

Fix  $t > 1$  sufficiently close to 1 and  $\beta > t$  satisfying  $\beta m < 1$ . Then, there exists  $C = C(\beta) > 0$  such that

$$\int_{\mathbb{R}} \left( e^{\omega|v_n(x,0)|^2} - 1 \right)^t dx \leq C \int_{\mathbb{R}} \left( e^{\beta m \omega \left( \frac{|v_n(x,0)|}{\|v_n\|} \right)^2} - 1 \right) dx,$$

for every  $n \geq n_0$ . Hence, by Lemma 2.1,

$$\int_{\mathbb{R}} \left( e^{\omega|v_n(x,0)|^2} - 1 \right)^t dx \leq C_1 \quad \forall n \geq n_0,$$

for some positive constant  $C_1$ . Now, the lemma follows fixing

$$C = \max \left\{ C_1, \int_{\mathbb{R}} \left( e^{\omega|v_1|^2} - 1 \right)^t dx, \dots, \int_{\mathbb{R}} \left( e^{\omega|v_{n_0}|^2} - 1 \right)^t dx \right\}.$$

■

**Corollary 2.3.** *Let  $(v_n)$  be a sequence in  $X^1(\mathbb{R}_+^2)$  satisfying (2.3). If  $v_n \rightharpoonup v$  weakly in  $X^1(\mathbb{R}_+^2)$  and  $v_n(x,0) \rightarrow v(x,0)$  a.e in  $\mathbb{R}$ , as  $n \rightarrow \infty$ , then,*

$$(2.4) \quad F(v_n(x,0)) \rightarrow F(v(x,0)) \quad \text{in } L^1(-R, R),$$

$$(2.5) \quad f(v_n(x,0))v_n(x,0) \rightarrow f(v(x,0))v(x,0) \quad \text{in } L^1(-R, R)$$

and

$$(2.6) \quad \int_{-R}^R f(v_n(x,0))\phi(x,0) dx \rightarrow \int_{-R}^R f(v(x,0))\phi(x,0) dx,$$

as  $n \rightarrow \infty$ , for all  $\phi \in X^1(\mathbb{R}_+^2)$  and  $R > 0$ .

**Proof.** By  $(f_1)$ , for each  $\beta > 1$  and  $\alpha > \alpha_0$ , there is  $C > 0$  such that

$$|F(s)| \leq C(|s|^2 + (e^{\alpha\beta|s|^2} - 1)) \quad \forall s \in \mathbb{R},$$

from where it follows that,

$$(2.7) \quad |F(v_n(x,0))| \leq C(|v_n(x,0)|^2 + (e^{\alpha\beta|v_n(x,0)|^2} - 1)), \quad \forall n \in \mathbb{N}.$$

Setting

$$h_n(x) = C(e^{\alpha_0\beta|v_n(x,0)|^2} - 1),$$

we can fix  $\beta, q > 1$  sufficiently close to 1 and  $\alpha$  sufficiently close to  $\alpha_0$  such that

$$h_n \in L^q(\mathbb{R}) \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|h_n\|_q < +\infty,$$

which is an immediate consequence of Lemma 2.2. Therefore, up to subsequence, we derive that

$$h_n \rightharpoonup h = C(e^{\alpha_0\beta|v(x,0)|^2} - 1) \quad \text{weakly in } L^q(\mathbb{R}), \text{ as } n \rightarrow \infty.$$

Since  $h_n, h \geq 0$ , the last limit yields

$$h_n \rightarrow h \quad \text{in } L^1(-R, R), \quad \forall R > 0, \text{ as } n \rightarrow \infty.$$

On the other hand, we know that

$$v_n(\cdot, 0) \rightarrow v(\cdot, 0) \quad \text{in } L^2(-R, R), \text{ as } n \rightarrow \infty.$$

Gathering the above limits with (2.7), we get

$$F(v_n(x,0)) \rightarrow F(v(x,0)) \quad \text{in } L^1(-R, R), \quad \forall R > 0, \text{ as } n \rightarrow \infty.$$

The limits (2.5) and (2.6) follow with the same type of arguments.  $\blacksquare$

The next lemma is a Lions type result, which is crucial in our approach. Since it follows with the same arguments found in C. Alves, J. M. do Ó and O. Miyagaki [2, Proposition 2.3], we will omit its proof.

**Lemma 2.4.** *Let  $(v_n) \subset X^1(\mathbb{R}_+^2)$  be a sequence with*

$$\limsup_{n \rightarrow +\infty} \|v_n\|^2 < 1.$$

*If there is  $R > 0$  such that*

$$\lim_{n \rightarrow +\infty} \sup_{z \in \mathbb{R}} \int_{z-R}^{z+R} |v_n(x, 0)|^2 dx = 0,$$

*then*

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}} F(v_n(x, 0)) dx = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}} f(v_n(x, 0)) v_n(x, 0) dx = 0.$$

### 3. THE CAFFARELLI AND SILVESTRE'S METHOD

First of all, using the change variable  $u(x) = v(\varepsilon x)$ , it is possible to prove that Problem  $(P_\varepsilon)$  is equivalent to the problem

$$(P'_\varepsilon) \quad \begin{cases} (-\Delta)^{1/2} u + V(\varepsilon x)u = f(u) & \text{in } \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}), \quad u > 0 & \text{on } \mathbb{R}. \end{cases}$$

Hereafter, to get a solution for  $(P'_\varepsilon)$ , we will use a method due to L. Caffarelli and L. Silvestre in [10], more exactly, due to R. Frank and E. Lenzmann [22] for whole line. In the seminal above papers, were developed a local interpretation of the fractional Laplacian given in  $\mathbb{R}$  by considering a Dirichlet to Neumann type operator in the domain  $\mathbb{R}_+^2 = \{(x, t) \in \mathbb{R}^2 : t > 0\}$ . A similar extension, in a bounded domain, see for instance, [6, 8, 9]. For  $u \in H^{1/2}(\mathbb{R})$ , the solution  $w \in X^1(\mathbb{R}_+^2)$  of

$$(3.1) \quad \begin{cases} -\operatorname{div}(\nabla w) = 0 & \text{in } \mathbb{R}_+^2 \\ w = u & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

is called 1/2-harmonic extension  $w = E_{1/2}(u)$  of  $u$  and it is proved in [10] that

$$\lim_{y \rightarrow 0^+} \frac{\partial w}{\partial y}(x, y) = -(-\Delta)^{1/2} u(x).$$

To get a solution for the nonlocal Problem  $(P'_\varepsilon)$ , we will study the existence of solution for the local problem defined on the upper half plane

$$(LP_\varepsilon) \quad \begin{cases} -\operatorname{div}(\nabla w) = 0 & \text{in } \mathbb{R}_+^2 \\ -\frac{\partial w}{\partial \nu} = -V(\varepsilon x)w + f(w) & \text{on } \mathbb{R} \times \{0\}, \end{cases}$$

where

$$\frac{\partial w}{\partial \nu} = \lim_{y \rightarrow 0^+} \frac{\partial w}{\partial y}(x, y),$$

since if  $w$  is a solution for the above problem, the function  $u(x) = w(x, 0)$  is a solution for  $(P'_\varepsilon)$ .

Associated with  $(LP_\varepsilon)$ , we have the  $J_\varepsilon : X^1(\mathbb{R}_+^2) \rightarrow \mathbb{R}$  defined by

$$(3.2) \quad J_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}_+^2} |\nabla v|^2 dx dy + \frac{1}{2} \int_{\mathbb{R}} V(\varepsilon x) |v(x, 0)|^2 dx - \int_{\mathbb{R}} F(v(x, 0)) dx,$$

which is  $C^1(X^1(\mathbb{R}_+^2), \mathbb{R})$  with derivative given by

$$(3.3) \quad \begin{aligned} J'_\varepsilon(v)\phi &= \frac{1}{2} \int_{\mathbb{R}_+^2} \nabla v \cdot \nabla \phi dx dy \\ &+ \frac{1}{2} \int_{\mathbb{R}} V(\varepsilon x) v(x, 0) \phi(x, 0) dx - \int_{\mathbb{R}} f(v(x, 0)) \phi(x, 0) dx, \quad \forall \phi \in X^1(\mathbb{R}_+^2). \end{aligned}$$

We would like point out that  $u$  is a solution of  $(P'_\varepsilon)$  if, and only if,  $u = v(x, 0)$  for all  $x \in \mathbb{R}$ , for some critical point  $v$  of  $J_\varepsilon$ .

In what follows, we will not work directly with functional  $J_\varepsilon$ , because we have some difficulties to prove that it verifies the  $(PS)$  condition. Hereafter, we will use the same approach explored in [16], modifying the nonlinearity of a suitable way. The idea is the following:

First of all, without loss of generality, we will assume that

$$(3.4) \quad 0 \in \Lambda \quad \text{and} \quad V(0) = V_0 = \inf_{x \in \mathbb{R}} V(x).$$

We recall that in assumption  $(f1)$  we imposed that  $f(t) = 0$ ,  $\forall t \leq 0$ , because we are looking for positive solutions. Moreover, let us choose  $k > 2\vartheta/(\vartheta - 2)$  and  $a > 0$  verifying

$$\frac{f(a)}{a} = \frac{V_0}{k},$$

where  $V_0 > 0$  was given in  $(V_1)$ . Using these numbers, we set the functions

$$\tilde{f}(t) = \begin{cases} f(t), & t \leq a, \\ \frac{V_0}{k} t, & t \geq a \end{cases}$$

and

$$g(x, t) = \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda) \tilde{f}(t), \quad \forall (x, t) \in \mathbb{R}^2,$$

where  $\Lambda$  was given in  $(V_2)$  and  $\chi_\Lambda$  denotes the characteristic function associated with  $\Lambda$ , that is,

$$\chi(x) = \begin{cases} 1, & x \in \Lambda, \\ 0, & x \in \Lambda^c. \end{cases}$$

Using the above functions, we will study the existence of positive solution for the following problem

$$(AP) \quad \begin{cases} (-\Delta)^{1/2} u + V(\varepsilon x) u = g_\varepsilon(x, u), & x \in \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}), \end{cases}$$

where

$$g_\varepsilon(x, t) = g(\varepsilon x, t), \quad \forall (x, t) \in \mathbb{R}^2.$$

We recall by using [10], to get a solution for the above problem, it is enough to study the existence of solution for the problem

$$(AP)' \quad \begin{cases} -\operatorname{div}(\nabla w) = 0 & \text{in } \mathbb{R}_+^2 \\ \frac{\partial w}{\partial \nu} = V(\varepsilon x) w - g_\varepsilon(x, w) & \text{on } \mathbb{R} \times \{0\}, \end{cases}$$

because if  $w$  is a solution of  $(AP)'$ , the function  $u(x) = w(x, 0)$  is a solution for  $(AP)$ .

Here, we would like point out that if  $v_\varepsilon \in X^1(\mathbb{R}_+^2)$  is a solution of  $(AP)'$  with

$$v_\varepsilon(x, 0) < a, \quad \forall x \in \Lambda_\varepsilon^c,$$

where  $\Lambda_\varepsilon = \Lambda/\varepsilon$ , then  $u_\varepsilon(x) = v_\varepsilon(x, 0)$  is a solution of  $(P'_\varepsilon)$ .

Associated with  $(AP)'$ , we have the energy functional  $E_\varepsilon : X^1(\mathbb{R}_+^2) \rightarrow \mathbb{R}$  given by

$$E_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}_+^2} |\nabla v|^2 dx dy + \frac{1}{2} \int_{\mathbb{R}} V(\varepsilon x) |v(x, 0)|^2 dx - \int_{\mathbb{R}} G_\varepsilon(x, v(x, 0)) dx$$

where

$$G_\varepsilon(x, t) = \int_0^t g_\varepsilon(x, \tau) d\tau, \quad \forall (x, t) \in \mathbb{R}^2.$$

Using the definition of  $g$ , it follows that

$$(g_1) \quad \vartheta G_\varepsilon(x, t) \leq g_\varepsilon(x, t)t, \quad \forall (x, t) \in \Lambda_\varepsilon \times \mathbb{R},$$

$$(g_2) \quad 2G_\varepsilon(x, t) \leq g_\varepsilon(x, t)t \leq \frac{V_0}{k} |t|^2, \quad \forall (x, t) \in (\Lambda_\varepsilon)^c \times \mathbb{R}.$$

From assumption  $(g_2)$ ,

$$(g_3) \quad L(x, t) = V(x) - G_\varepsilon(x, t) \geq \left(1 - \frac{1}{2k}\right) V(x) |t|^2 \geq 0, \quad \forall (x, t) \in (\Lambda_\varepsilon)^c \times \mathbb{R},$$

$$(g_4) \quad M(x, t) = V(x) - g_\varepsilon(x, t)t \geq \left(1 - \frac{1}{k}\right) V(x) |t|^2 \geq 0, \quad \forall (x, t) \in (\Lambda_\varepsilon)^c \times \mathbb{R}.$$

**Lemma 3.1.** *The functional  $E_\varepsilon$  verifies the mountain pass geometry, that is,*

*i) There are  $r, \rho > 0$  such that*

$$E_\varepsilon(v) \geq \rho, \quad \text{for } \|v\| = r$$

*ii) There is  $e \in X^1(\mathbb{R}_+^2)$  with  $\|e\| > r$  and  $E_\varepsilon(e) < 0$ .*

**Proof.** From  $(g_1) - (g_4)$ , there exist  $c_1, c_2 > 0$  verifying

$$E_\varepsilon(v) \geq c_1 \|v\|^2 - c_2 \|v\|^q, \quad \forall v \in X^1(\mathbb{R}_+^2).$$

From the above inequality, there are  $r, \rho > 0$  such that

$$E_\varepsilon(v) \geq \rho, \quad \text{for } \|v\|_{1,s} = r,$$

showing *i*). To prove *ii*), fix  $\varphi \in X^1(\mathbb{R}_+^2)$  with  $\text{supp } \varphi \subset \Lambda_\varepsilon \times \mathbb{R}$ . Then, for  $t > 0$

$$E_\varepsilon(t\varphi) = \frac{t^2}{2} \|\varphi\|^2 - \int_{\mathbb{R}} F(t\varphi(x, 0)) dx.$$

From  $(f_3)$ , we know that there are  $c_3, c_4 \geq 0$  verifying

$$F(t) \geq c_1 |t|^\vartheta - c_2, \quad \forall t \geq 0.$$

Using the above inequality, we derive

$$\lim_{t \rightarrow +\infty} E_\varepsilon(t\varphi) = -\infty.$$

Thereby, *ii*) follows with  $e = t\varphi$  and  $t$  large enough. ■



In what follows, we denote by  $c_\varepsilon$  the mountain pass level associated with  $E_\varepsilon$ . Related to the case  $\varepsilon = 0$ , it is possible to prove that there is  $w_0 \in X^1(\mathbb{R}_+^2)$  such that

$$(3.5) \quad J_0(w_0) = c_0 \quad \text{and} \quad J'_0(w_0) = 0.$$

The existence of  $w_0$  can be obtained repeating the same approach explored in [2].

**Lemma 3.2.** *The minimax level  $c_0$  verifies*

$$0 < c_0 < \min\{1, V_0\} \left( \frac{1}{2} - \frac{1}{\vartheta} \right).$$

**Proof.** Consider  $w_* \in X^1(\mathbb{R}_+^2)$  verifying

$$\tilde{J}_0(w_*) = \beta_p \quad \text{and} \quad \tilde{J}'_0(w_*) = 0.$$

By characterization of  $c_0$ ,

$$c_0 \leq \max_{t \geq 0} J_0(tw_*).$$

Consequently, by  $(f_5)$ ,

$$c_0 \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \int_{\mathbb{R}_+^2} |\nabla w_*|^2 \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}} V_0 |w_*(x, 0)|^2 \, dx - \frac{C_p t^p}{p} \int_{\mathbb{R}} |w_*(x, 0)|^p \, dx \right\},$$

which implies that

$$c_0 \leq C_p^{2/(2-p)} \beta_p.$$

Hence, from  $(f_5)$ ,

$$0 < c_0 < \min\{1, V_0\} \left( \frac{1}{2} - \frac{1}{\vartheta} \right).$$

■

Hereafter, we will assume that  $k$  is large enough such that

$$0 < c_0 < \min\{1, V_0\} \left( \left( \frac{1}{2} - \frac{1}{\vartheta} \right) - \frac{1}{k} \right) < \min\{1, V_0\} \left( \frac{1}{2} - \frac{1}{\vartheta} \right).$$

The next lemma establishes an important relation between  $c_\varepsilon$  and  $c_0$ .

**Lemma 3.3.** *The numbers  $c_0$  and  $c_\varepsilon$  verify the equality below*

$$(3.6) \quad \lim_{\varepsilon \rightarrow 0} c_\varepsilon = c_0.$$

Hence, there is  $\varepsilon_0 > 0$  such that

$$(3.7) \quad 0 < \sup_{\varepsilon \in (0, \varepsilon_0)} c_\varepsilon < \min\{1, V_0\} \left( \left( \frac{1}{2} - \frac{1}{\vartheta} \right) - \frac{1}{k} \right).$$

**Proof.** From  $(V_1)$ ,

$$c_\varepsilon \geq c_0, \quad \forall \varepsilon \geq 0.$$

Then,

$$(3.8) \quad \liminf_{\varepsilon \rightarrow 0} c_\varepsilon \geq c_0.$$

Next, fix  $t_\varepsilon > 0$  such that

$$t_\varepsilon w \in \mathcal{M}_\varepsilon = \{v \in X^1(\mathbb{R}_+^2) \setminus \{0\} : E'_\varepsilon(v)v = 0\}.$$

By definition of  $c_\varepsilon$ , we know that

$$c_\varepsilon \leq \max_{t \geq 0} E_\varepsilon(tw) = E_\varepsilon(t_\varepsilon w).$$

Now standard arguments as those used in [19], it is possible to prove that

$$\lim_{\varepsilon \rightarrow 0} t_\varepsilon = 1$$

and

$$\lim_{\varepsilon \rightarrow 0} E_\varepsilon(t_\varepsilon w) = J_0(w).$$

Thus,

$$(3.9) \quad \limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq J_0(w) = c_0.$$

From (3.8) and (3.9),

$$\limsup_{\varepsilon \rightarrow 0} c_\varepsilon = c_0,$$

showing (3.6). The inequality (3.7) is an immediate consequence of (3.6) and Lemma 3.2.  $\blacksquare$

**Lemma 3.4.** *Let  $\varepsilon \in (0, \varepsilon_0)$  and  $(v_n) \subset X^1(\mathbb{R}_+^2)$  be a  $(PS)_{c_\varepsilon}$  sequence for  $E_\varepsilon$ . Then,*

$$\limsup_{n \rightarrow +\infty} \|v_n\|^2 < 1.$$

**Proof.** Gathering  $E_\varepsilon(u_n) - \frac{1}{\vartheta} E'_\varepsilon(u_n)u_n = c_\varepsilon + o_n(1)$  with definition of  $g$ , we find

$$\left(\frac{1}{2} - \frac{1}{\vartheta}\right) \int_{\mathbb{R}_+^2} |\nabla v_n|^2 dx dy + \left(\left(\frac{1}{2} - \frac{1}{\vartheta}\right) - \frac{1}{k}\right) V_0 \int_{\mathbb{R}} |v_n(x, 0)|^2 dx \leq c_\varepsilon + o_n(1),$$

from where it follows that

$$\min\{1, V_0\} \left(\left(\frac{1}{2} - \frac{1}{\vartheta}\right) - \frac{1}{k}\right) \limsup_{n \rightarrow +\infty} \|v_n\|^2 \leq c_\varepsilon < \min\{1, V_0\} \left(\left(\frac{1}{2} - \frac{1}{\vartheta}\right) - \frac{1}{k}\right),$$

and so,

$$\limsup_{n \rightarrow +\infty} \|v_n\|^2 < 1. \quad \blacksquare$$

**Lemma 3.5.** *For  $\varepsilon \in (0, \varepsilon_0)$ , the functional  $E_\varepsilon$  verifies the  $(PS)_{c_\varepsilon}$  condition.*

**Proof.** Let  $(v_n) \subset X^1(\mathbb{R}_+^2)$  be a  $(PS)_{c_\varepsilon}$  sequence for  $E_\varepsilon$ , that is,

$$E_\varepsilon(v_n) \rightarrow c_\varepsilon \quad \text{and} \quad E'_\varepsilon(v_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

From Lemma 3.4,  $(v_n)$  is bounded in  $X^1(\mathbb{R}_+^2)$  and

$$\limsup_{n \rightarrow +\infty} \|v_n\|^2 < 1.$$

Since  $X^1(\mathbb{R}_+^2)$  is reflexive, there is a subsequence of  $(v_n)$ , still denoted by itself, and  $v \in X^1(\mathbb{R}_+^2)$  such that

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{weakly in } X^1(\mathbb{R}_+^2), \text{ as } n \rightarrow \infty, \\ v_n &\rightarrow v \quad \text{in } L_{loc}^q(\mathbb{R}), \quad \forall q \in [2, +\infty), \text{ as } n \rightarrow \infty, \end{aligned}$$

and

$$v_n(x, 0) \rightarrow v(x, 0) \quad \text{a.e. in } \mathbb{R}, \text{ as } n \rightarrow \infty.$$

Moreover, by Lemma 2.3,

$$\int_{\mathbb{R}} f(v_n(x, 0))\phi(x, 0) dx \rightarrow \int_{\mathbb{R}} f(v(x, 0))\phi(x, 0) dx,$$

as  $n \rightarrow \infty$ , for all  $\phi \in C_0^\infty(\overline{\mathbb{R}_+^2})$ .

Using the above limits, it is possible to prove that  $v$  is a critical point for  $E_\varepsilon$ , that is,

$$E'_\varepsilon(v)\varphi = 0, \quad \forall \varphi \in X^1(\mathbb{R}_+^2).$$

Considering  $\varphi = v$ , we have that  $E'_\varepsilon(v)v = 0$ , and so,

$$\begin{aligned} & \int_{\mathbb{R}_+^2} |\nabla v|^2 dx dy + \int_{\Lambda_\varepsilon} V(\varepsilon x) |v(x, 0)|^2 dx + \int_{(\Lambda_\varepsilon)^c} M(x, v(x, 0)) dx \\ &= \int_{\Lambda_\varepsilon} f(v(x, 0))v(x, 0) dx. \end{aligned}$$

On the other hand, using the limit  $E'_\varepsilon(v_n)v_n = o_n(1)$ , we derive that

$$\begin{aligned} & \int_{\mathbb{R}_+^2} |\nabla v_n|^2 dx dy + \int_{\Lambda_\varepsilon} V(\varepsilon x) |v_n(x, 0)|^2 dx + \int_{(\Lambda_\varepsilon)^c} M(x, v_n(x, 0)) dx \\ &= \int_{\Lambda_\varepsilon} f(v_n(x, 0))v_n(x, 0) dx + o_n(1). \end{aligned}$$

Since  $\Lambda_\varepsilon$  is bounded, by the compactness of sobolev embedding and Lemma 2.3 yield

$$\lim_{n \rightarrow +\infty} \int_{\Lambda_\varepsilon} f(v_n(x, 0))v_n(x, 0) dx = \int_{\Lambda_\varepsilon} f(v(x, 0))v(x, 0) dx$$

and

$$(3.10) \quad \lim_{n \rightarrow +\infty} \int_{\Lambda_\varepsilon} V(\varepsilon x) |v_n(x, 0)|^2 dx = \int_{\Lambda_\varepsilon} V(\varepsilon x) |v(x, 0)|^2 dx.$$

Therefore,

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \left( \int_{\mathbb{R}_+^2} |\nabla v_n|^2 dx dy + \int_{(\Lambda_\varepsilon)^c} M(x, v_n(x, 0)) dx \right) \\ &= \int_{\mathbb{R}_+^2} |\nabla v|^2 dx dy + \int_{(\Lambda_\varepsilon)^c} M(x, v(x, 0)) dx. \end{aligned}$$

Now, recalling that  $M(x, t) \geq 0$ , the Fatous' lemma leads

$$\begin{aligned} & \liminf_{n \rightarrow +\infty} \left( \int_{\mathbb{R}_+^2} |\nabla v_n|^2 dx dy + \int_{(\Lambda_\varepsilon)^c} M(x, v_n(x, 0)) dx \right) \\ &\geq \int_{\mathbb{R}_+^2} |\nabla v|^2 dx dy + \int_{(\Lambda_\varepsilon)^c} M(x, v(x, 0)) dx \end{aligned}$$

Hence,

$$(3.11) \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{R}_+^2} |\nabla v_n|^2 dx dy = \int_{\mathbb{R}_+^2} |\nabla v|^2 dx dy$$

and

$$\lim_{n \rightarrow +\infty} \int_{(\Lambda_\varepsilon)^c} M(x, v(x, 0)) dx = \int_{(\Lambda_\varepsilon)^c} M(x, v(x, 0)) dx.$$

The last limit combined with definition of function  $M$  gives

$$\lim_{n \rightarrow +\infty} \int_{(\Lambda_\varepsilon)^c} V(\varepsilon x) |v_n(x, 0)|^2 dx = \int_{(\Lambda_\varepsilon)^c} V(\varepsilon x) |v(x, 0)|^2 dx.$$

Gathering this limit with (3.10), we deduce that

$$(3.12) \quad \lim_{n \rightarrow +\infty} \int_{\mathbb{R}} V(\varepsilon x) |v_n(x, 0)|^2 dx = \int_{\mathbb{R}} V(\varepsilon x) |v(x, 0)|^2 dx.$$

From (3.11)-(3.12),

$$\lim_{n \rightarrow +\infty} \|v_n\|_{\varepsilon}^2 = \|v\|_{\varepsilon}^2.$$

As  $X^1(\mathbb{R}_+^2)$  is a Hilbert space and  $v_n \rightharpoonup v$  weakly in  $X^1(\mathbb{R}_+^2)$ , as  $n \rightarrow \infty$ , the above limit yields

$$v_n \rightarrow v \quad \text{in} \quad X^1(\mathbb{R}_+^2), \quad \text{as } n \rightarrow \infty,$$

showing that  $E_{\varepsilon}$  verifies the  $(PS)_{c_{\varepsilon}}$ . ■

**Theorem 3.6.** *For  $\varepsilon \in (0, \varepsilon_0)$ , the functional  $E_{\varepsilon}$  has a nonnegative critical point  $v_{\varepsilon} \in X^1(\mathbb{R}_+^2)$  such*

$$(3.13) \quad E_{\varepsilon}(v_{\varepsilon}) = c_{\varepsilon} \quad \text{and} \quad E'_{\varepsilon}(v_{\varepsilon}) = 0.$$

**Proof.** From Lemma 3.3, there is  $\varepsilon_0 > 0$ , such that  $E_{\varepsilon}$  verifies the  $(PS)_{c_{\varepsilon}}$  condition for  $\varepsilon \in (0, \varepsilon_0)$ . Then, the existence of  $v_{\varepsilon}$  is an immediate consequence of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz (see e.g. [28]). The function  $v_{\varepsilon}$  is nonnegative, because

$$E'_{\varepsilon}(v_{\varepsilon})(v_{\varepsilon}^{-}) = 0 \implies v_{\varepsilon}^{-} = 0,$$

where  $v_{\varepsilon}^{-} = \min\{v_{\varepsilon}, 0\}$ . ■

**Lemma 3.7.** *Decreasing  $\varepsilon_0$ , if necessary, there are  $r, \beta > 0$  and  $(y_{\varepsilon}) \subset \mathbb{R}$  such that*

$$(3.14) \quad \int_{y_{\varepsilon}-r}^{y_{\varepsilon}+r} |v_{\varepsilon}(x, 0)|^2 dx \geq \beta, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

**Proof.** First of all, we recall that since  $(v_{\varepsilon})$  satisfies (3.13), there is  $\alpha > 0$ , which is independent of  $\varepsilon$ , such that

$$(3.15) \quad \|v_{\varepsilon}\|_{\varepsilon}^2 \geq \alpha, \quad \forall \varepsilon > 0.$$

To show (3.14), it is enough to see that for any sequence  $(\varepsilon_n) \subset (0, +\infty)$  with  $\varepsilon_n \rightarrow 0$ , the limit below

$$\lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{y-r}^{y+r} |v_{\varepsilon_n}(x, 0)|^2 dx = 0,$$

does not hold for any  $r > 0$ . Otherwise, if it holds for some  $r > 0$ , by Lemma 2.4,

$$\int_{\mathbb{R}} f(v_{\varepsilon_n}(\cdot, 0)) v_n(x, 0) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

implying that

$$\|v_{\varepsilon_n}\|_{\varepsilon}^2 \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

which contradicts (3.15). ■

**Lemma 3.8.** *For any  $\varepsilon_n \rightarrow 0$ , consider the sequence  $(y_{\varepsilon_n}) \subset \mathbb{R}$  given in Lemma 3.7 and  $\psi_n(x, y) = v_{\varepsilon_n}(x + y_{\varepsilon_n}, y)$ . Then, up to subsequence, there is  $\psi \in X^1(\mathbb{R}_+^2)$  such that*

$$(3.16) \quad \psi_n \rightarrow \psi \quad \text{in} \quad X^1(\mathbb{R}_+^2), \quad \text{as } n \rightarrow \infty.$$

Moreover, there is  $x_0 \in \Lambda$  such that

$$(3.17) \quad \lim_{n \rightarrow 0} \varepsilon_n y_{\varepsilon_n} = x_0 \quad \text{and} \quad V(x_0) = V_0.$$

**Proof.** We begin the proof showing that  $(\varepsilon_n y_{\varepsilon_n})$  is a bounded sequence. Hereafter, we denote by  $(y_n)$  and  $(v_n)$  the sequences  $(y_{\varepsilon_n})$  and  $(v_{\varepsilon_n})$  respectively.

Since  $E'_{\varepsilon_n}(v_n)\phi = 0, \forall \phi \in X^1(\mathbb{R}_+^2)$ , we have that

$$\int_{\mathbb{R}_+^2} \nabla v_n \nabla \phi \, dx dy + \int_{\mathbb{R}} V(\varepsilon_n x) v_n(x, 0) \phi(x, 0) \, dx - \int_{\mathbb{R}} g_{\varepsilon}(x, v_n(x, 0)) \phi(x, 0) \, dx = 0.$$

Then,

$$\int_{\mathbb{R}_+^2} |\nabla v_n|^2 \, dx dy + \int_{\mathbb{R}} V(\varepsilon_n x) |v_n(x, 0)|^2 \, dx - \int_{\mathbb{R}} g_{\varepsilon}(x, v_n(x, 0)) v_n(x, 0) \, dx = 0.$$

From definition of  $g$ , we see that

$$g_{\varepsilon}(x, t) \leq f(t), \quad \forall t \geq 0,$$

and reminding that  $v_n \geq 0$ , we infer that

$$\int_{\mathbb{R}_+^2} |\nabla v_n|^2 \, dx dy + \int_{\mathbb{R}} V_0 |v_n(x, 0)|^2 \, dx - \int_{\mathbb{R}} f(v_n(x, 0)) v_n(x, 0) \, dx \leq 0.$$

Therefore, there is  $s_n \in (0, 1)$  such that

$$s_n v_n \in \mathcal{M}_0 = \{v \in X^1(\mathbb{R}_+^2) \setminus \{0\} : J'_0(v)v = 0\}.$$

Using the characterization of  $c_0$ , we know that

$$c_0 \leq J_0(s_n v_n), \quad \forall n \in \mathbb{N}.$$

As

$$J_0(w) \leq E_{\varepsilon}(w), \quad \forall w \in X^1(\mathbb{R}_+^2) \quad \text{and} \quad \varepsilon > 0,$$

it follows that

$$c_0 \leq J_0(s_n v_n) \leq E_{\varepsilon_n}(s_n v_n) \leq \max_{s \geq 0} E_{\varepsilon_n}(s v_n) = E_{\varepsilon_n}(v_n) = c_{\varepsilon_n}.$$

Recalling that

$$c_{\varepsilon_n} \rightarrow c_0, \quad \text{as } n \rightarrow \infty,$$

the last inequality gives

$$(s_n v_n) \subset \mathcal{M}_0, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad J_0(s_n v_n) \rightarrow c_0, \quad \text{as } n \rightarrow \infty.$$

By change variable, we also have

$$(s_n \psi_n) \subset \mathcal{M}_0, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad J_0(s_n \psi_n) \rightarrow c_0, \quad \text{as } n \rightarrow \infty.$$

Using Ekeland Variational Principle, we can assume that  $(s_n v_n)$  is a  $(PS)_{c_0}$  sequence, that is,

$$(s_n \psi_n) \subset \mathcal{M}_0, \quad \forall n \in \mathbb{N}, \quad J_0(s_n \psi_n) \rightarrow c_0 \quad \text{and} \quad J'_0(s_n \psi_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

A direct computation shows that  $(s_n)$  is a bounded sequence with

$$\liminf_{n \rightarrow +\infty} s_n > 0.$$

Thus, in what follows, we can assume that for some subsequence, there is  $s_0 > 0$  such that

$$s_n \rightarrow s_0, \quad \text{as } n \rightarrow \infty.$$

From definition of  $y_n$  and  $\psi_n$ , we know that  $\psi \in X^1(\mathbb{R}_+^2) \setminus \{0\}$ . Moreover, as  $J'_0(s_n \psi_n) \rightarrow 0$ , we also have  $J'_0(s_0 \psi) = 0$ . Thereby, by definition of  $c_0$ , we obtain

$$c_0 \leq J_0(s_0 \psi).$$

On the other hand, by Fatou's Lemma we obtain

$$\liminf_{n \rightarrow +\infty} J_0(s_n \psi_n) \geq J_0(s_0 \psi)$$

which implies

$$J_0(s_0 \psi) = c_0 \quad \text{and} \quad J'_0(s_0 \psi) = 0.$$

The above equalities combined with Fatou's Lemma, up to a subsequence, gives

$$s_n \psi_n \rightarrow s_0 \psi \quad \text{in} \quad X^1(\mathbb{R}_+^2), \quad \text{as } n \rightarrow \infty.$$

Recalling that  $s_n \rightarrow s_0 > 0$ , as  $n \rightarrow \infty$ , we can conclude that

$$\psi_n \rightarrow \psi \quad \text{in} \quad X^1(\mathbb{R}_+^2), \quad \text{as } n \rightarrow \infty,$$

showing (3.16).

Using the last limit, we are able to prove (3.17). To do end, we begin making the following claim

**Claim 3.1.**  $\lim_{n \rightarrow +\infty} \text{dist}(\varepsilon_n y_n, \bar{\Lambda}) = 0$

Indeed, if the claim does not hold, there is  $\delta > 0$  and a subsequence of  $(\varepsilon_n y_n)$ , still denoted by itself, such that,

$$\text{dist}(\varepsilon_n y_n, \bar{\Lambda}) \geq \delta, \quad \forall n \in \mathbb{N}.$$

Consequently, there is  $r > 0$  such that

$$(\varepsilon_n y_n - r, \varepsilon_n y_n + r) \subset \Lambda^c, \quad \forall n \in \mathbb{N}.$$

From definition of  $\psi_n$ , we have that

$$\begin{aligned} & \int_{\mathbb{R}_+^2} |\nabla \psi_n|^2 \, dx dy + \int_{\mathbb{R}} V(\varepsilon_n x + \varepsilon_n y_n) |\psi_n(x, 0)|^2 \, dx \\ &= \int_{\mathbb{R}} g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x, 0)) \psi_n(x, 0) \, dx. \end{aligned}$$

Note that

$$\begin{aligned} & \int_{\mathbb{R}} g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x, 0)) \psi_n(x, 0) \, dx \leq \int_{-\frac{r}{\varepsilon_n}}^{\frac{r}{\varepsilon_n}} g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x, 0)) \psi_n(x, 0) \, dx \\ & + \left( \int_{-\infty}^{-\frac{r}{\varepsilon_n}} + \int_{\frac{r}{\varepsilon_n}}^{+\infty} \right) g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x, 0)) \psi_n(x, 0) \, dx, \end{aligned}$$

and so,

$$\begin{aligned} & \int_{\mathbb{R}} g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x, 0)) \psi_n(x, 0) \, dx \\ & \leq \frac{V_0}{k} \int_{-\frac{r}{\varepsilon_n}}^{\frac{r}{\varepsilon_n}} |\psi_n(x, 0)|^2 \, dx + \left( \int_{-\infty}^{-\frac{r}{\varepsilon_n}} + \int_{\frac{r}{\varepsilon_n}}^{+\infty} \right) f(\psi_n(x, 0)) \psi_n(x, 0) \, dx. \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_{\mathbb{R}_+^2} |\nabla \psi_n|^2 \, dx dy + \int_{\mathbb{R}} V(\varepsilon_n x + \varepsilon_n y_n) |\psi_n(x, 0)|^2 \, dx \\ & \leq \frac{V_0}{k} \int_{-\frac{r}{\varepsilon_n}}^{\frac{r}{\varepsilon_n}} |\psi_n(x, 0)|^2 \, dx + \left( \int_{-\infty}^{-\frac{r}{\varepsilon_n}} + \int_{\frac{r}{\varepsilon_n}}^{+\infty} \right) f(\psi_n(x, 0)) \psi_n(x, 0) \, dx. \end{aligned}$$

implying that

$$(3.18) \quad \int_{\mathbb{R}_+^2} |\nabla \psi_n|^2 dx dy + A \int_{\mathbb{R}^N} |\psi_n(x, 0)|^2 dx \leq \left( \int_{-\infty}^{-\frac{r}{\varepsilon_n}} + \int_{\frac{r}{\varepsilon_n}}^{+\infty} \right) f(\psi_n(x, 0)) \psi_n(x, 0) dx,$$

where  $A = V_0 \left(1 - \frac{1}{k}\right)$ . By (3.16),

$$\left( \int_{-\infty}^{-\frac{r}{\varepsilon_n}} + \int_{\frac{r}{\varepsilon_n}}^{+\infty} \right) f(\psi_n(x, 0)) \psi_n(x, 0) dx \rightarrow 0, \text{ as } n \rightarrow \infty,$$

and as  $n \rightarrow \infty$ ,

$$\int_{\mathbb{R}_+^2} |\nabla \psi_n|^2 dx dy + A \int_{\mathbb{R}} |\psi_n(x, 0)|^2 dx \rightarrow \int_{\mathbb{R}_+^2} |\nabla \psi|^2 dx dy + A \int_{\mathbb{R}} |\psi(x, 0)|^2 dx > 0,$$

which contradicts (3.18). This proves Claim 3.1.

From Claim 3.1, there are a subsequence of  $(\varepsilon_n y_n)$  and  $x_0 \in \overline{\Lambda}$  such that

$$\lim_{n \rightarrow +\infty} \varepsilon_n y_n = x_0.$$

**Claim 3.2.**  $x_0 \in \Lambda$ .

Indeed, from definition of  $\psi_n$ ,

$$\int_{\mathbb{R}_+^2} |\nabla \psi_n|^2 dx dy + \int_{\mathbb{R}} V(\varepsilon_n x + \varepsilon_n y_n) |\psi_n(x, 0)|^2 dx \leq \int_{\mathbb{R}} f(\psi_n(x, 0)) \psi_n(x, 0) dx.$$

Then, by (3.16),

$$\int_{\mathbb{R}_+^2} |\nabla \psi|^2 dx dy + \int_{\mathbb{R}} V(x_0) |\psi(x, 0)|^2 dx \leq \int_{\mathbb{R}} f(\psi(x, 0)) \psi(x, 0) dx.$$

Hence, there is  $s_1 \in (0, 1)$  such that

$$s_1 \psi \in \mathcal{M}_{V(x_0)} = \left\{ v \in X^1(\mathbb{R}_+^2) \setminus \{0\} : \tilde{J}'_{V(x_0)} v = v \right\}$$

where  $\tilde{J}_{V(x_0)} : X^1(\mathbb{R}_+^2) \rightarrow \mathbb{R}$  is given by

$$\tilde{J}_{V(x_0)}(v) = \frac{1}{2} \int_{\mathbb{R}_+^2} |\nabla v|^2 dx dy + \frac{1}{2} \int_{\mathbb{R}} V(x_0) |v(x, 0)|^2 dx - \int_{\mathbb{R}} F(v(x, 0)) dx.$$

If  $\tilde{c}_{V(x_0)}$  denotes the mountain pass level associated with  $\tilde{J}_{V(x_0)}$ , we must have

$$\tilde{c}_{V(x_0)} \leq \tilde{J}_{V(x_0)}(s_1 \psi) \leq \liminf_{n \rightarrow +\infty} E_{\varepsilon_n}(v_n) = \liminf_{n \rightarrow +\infty} c_{\varepsilon_n} = c_0 = \tilde{c}_{V(0)}.$$

Hence,

$$\tilde{c}_{V(x_0)} \leq \tilde{c}_{V(0)},$$

from where it follows that

$$V(x_0) \leq V(0).$$

As  $V_0 = \inf_{x \in \mathbb{R}} V(x)$ , the above inequality implies that

$$V(x_0) = V(0) = V_0.$$

Moreover, by  $(V_2)$ ,  $x_0 \notin \partial \Lambda$ . Then,  $x_0 \in \Lambda$ , finishing the proof. ■

**Corollary 3.9.** *Let  $(\psi_n)$  the sequence given in Lemma 3.8. Then,  $\psi_n(\cdot, 0) \in L^\infty(\mathbb{R})$  and there is  $K > 0$  such that*

$$(3.19) \quad |\psi_n(\cdot, 0)|_\infty \leq K, \quad \forall n \in \mathbb{N}$$

and

$$(3.20) \quad \psi_n(\cdot, 0) \rightarrow \psi(\cdot, 0) \quad \text{in } L^p(\mathbb{R}), \quad \forall p \in (2, +\infty), \text{ as } n \rightarrow \infty.$$

As an immediate consequence, the sequence  $h_n(x) = g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x, 0))$  must verify

$$(3.21) \quad h_n \rightarrow f(\psi(\cdot, 0)) \quad \text{in } L^p(\mathbb{R}), \quad \forall p \in (2, +\infty), \text{ as } n \rightarrow \infty.$$

**Proof.** In what follows, for each  $L > 0$ , we set

$$\psi_{n,L}(x, y) = \begin{cases} \psi_n(x, y), & \text{if } \psi_n(x, y) \leq L \\ L, & \text{if } \psi_n(x, y) \geq L \end{cases}$$

and

$$z_{n,L} = \psi_{n,L}^{2(\beta-1)} \psi_n,$$

with  $\beta > 1$  to be determined later. Since

$$\begin{aligned} & \int_{\mathbb{R}_+^2} \nabla \psi_n \nabla \phi \, dx dy + \int_{\mathbb{R}} V(\varepsilon_n x + \varepsilon_n y_n) \psi_n(x, 0) \phi(x, 0) \, dx \\ & - \int_{\mathbb{R}} g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x, 0)) \phi(x, 0) \, dx = 0, \quad \forall \phi \in X^1(\mathbb{R}_+^2), \quad \forall n \in \mathbb{N}, \end{aligned}$$

adapting the same approach explored in C. Alves and G. Figueiredo [3, Lemma 4.1] and using the fact that  $(\psi_n)$  is bounded in  $X^1(\mathbb{R}_+^2)$ , we conclude that there is  $K > 0$  such that

$$|\psi_n(\cdot, 0)|_\infty \leq K, \quad \forall n \in \mathbb{N}.$$

Now, the limit (3.20) is obtained by interpolation on the  $L^p$  spaces, while that (3.21) follows combining the growth condition on  $g$  with (3.20).  $\blacksquare$

In what follows, we denote by  $(w_n) \subset H^{1/2}(\mathbb{R})$  the sequence  $(\psi_n(\cdot, 0))$ , that is,

$$w_n(x) = \psi_n(x, 0), \quad \forall x \in \mathbb{R}.$$

Since

$$\begin{aligned} & \int_{\mathbb{R}_+^2} \nabla \psi_n \nabla \phi \, dx dy + \int_{\mathbb{R}} V(\varepsilon_n x + \varepsilon_n y_n) \psi_n(x, 0) \phi(x, 0) \, dx \\ & - \int_{\mathbb{R}} g(\varepsilon_n x + \varepsilon_n y_n, \psi_n(x, 0)) \phi(x, 0) \, dx = 0, \quad \forall \phi \in X^1(\mathbb{R}_+^2), \end{aligned}$$

we have that  $w_n$  is a solution of the problem

$$(-\Delta)^{1/2} w_n + V(\varepsilon_n x + \varepsilon_n y_n) w_n = g(\varepsilon_n x + \varepsilon_n y_n, w_n), \quad \text{in } \mathbb{R},$$

or equivalently,

$$(3.22) \quad (-\Delta)^{1/2} u + w_n = \chi_n, \quad \text{in } \mathbb{R},$$

where

$$(3.23) \quad \chi_n(x) = w_n(x) + g(\varepsilon_n x + \varepsilon_n y_n, w_n(x)) - V(\varepsilon_n x + \varepsilon_n y_n) w_n(x), \quad x \in \mathbb{R}.$$

Denoting  $\chi(x) = w(x) + f(w(x)) - V(x_0)w(x)$ , by Corollary 3.9, we have that

$$(3.24) \quad \chi_n \rightarrow \chi \quad \text{in } L^p(\mathbb{R}), \quad \forall p \in [2, +\infty), \text{ as } n \rightarrow \infty,$$



and there is  $k_1 > 0$ ,

$$(3.25) \quad |\chi_n|_\infty \leq k_1, \quad \forall n \in \mathbb{N}.$$

Motivated by some results found in [7] ( see also [21] ), which holds for whole line, we deduce that

$$w_n(x) = (\mathcal{K} * \chi_n)(x) = \int_{\mathbb{R}} \mathcal{K}(x-y) \chi_n(y) \, dy,$$

where  $\mathcal{K}$  is the Bessel kernel, which verifies:

( $K_1$ )  $\mathcal{K}$  is positive and even on  $\mathbb{R} \setminus \{0\}$ ,

( $K_2$ ) There is  $C > 0$  such that

$$\mathcal{K}(x) \leq C/|x|^2, \quad \forall x \in \mathbb{R} \setminus \{0\}$$

and

( $K_3$ )  $\mathcal{K} \in L^q(\mathbb{R})$ ,  $\forall q \in L^q(\mathbb{R}) \quad \forall q \in [1, \infty]$ .

Using the above informations, we are able to prove the following result

**Lemma 3.10.** *The sequence  $(w_n)$  verifies*

$$w_n(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty,$$

uniformly in  $n \in \mathbb{N}$ .

**Proof.** Given  $\delta > 0$ , we have

$$\begin{aligned} 0 \leq w_n(x) &\leq \int_{\mathbb{R}} \mathcal{K}(x-y) |\chi_n|(y) \, dy \\ &= \left( \int_{-\infty}^{x-1/\delta} + \int_{x+1/\delta}^{+\infty} \right) \mathcal{K}(x-y) |\chi_n|(y) \, dy + \int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y) |\chi_n|(y) \, dy \end{aligned}$$

from ( $K_2$ ), we have that, for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} (3.26) \quad \left( \int_{-\infty}^{x-1/\delta} + \int_{x+1/\delta}^{+\infty} \right) \mathcal{K}(x-y) |\chi_n|(y) \, dy &\leq C\delta^{1/2} |\chi_n|_\infty \left( \int_{-\infty}^{x-1/\delta} + \int_{x+1/\delta}^{+\infty} \right) \frac{dy}{|x-y|^{3/2}} \\ &\leq C\delta^{1/2} k_1 \left( \int_{-\infty}^{x-1} + \int_{x+1}^{+\infty} \right) \frac{dy}{|x-y|^{3/2}} = C_1 \delta^{1/2}. \end{aligned}$$

On the other hand,

$$\int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y) |\chi_n|(y) \, dy \leq \int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y) |\chi_n - \chi|(y) \, dy + \int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y) |\chi|(y) \, dy.$$

Fix  $q > 1$  with  $q$  sufficiently close to 1 and  $q' > 2$  such that  $1/q + 1/q' = 1$ . From ( $K_2$ ) and (3.22),

$$\int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y) |\chi_n|(y) \, dy \leq |\mathcal{K}|_q |\chi_n - \chi|_{q'} + |\mathcal{K}|_q |\chi|_{L^{q'}(x-1/\delta, x+1/\delta)}$$

As

$$|\chi_n - \chi|_{q'} \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty$$

and

$$|\chi|_{L^{q'}(x-1/\delta, x+1/\delta)} \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty,$$

we deduce that there are  $R > 0$  and  $n_0 \in \mathbb{N}$  such that

$$(3.27) \quad \int_{x-1/\delta}^{x+1/\delta} \mathcal{K}(x-y) |\chi_n|(y) \, dy \leq \delta, \quad \forall n \geq n_0 \quad \text{and} \quad |x| \geq R.$$

from (3.26) and (3.27),

$$(3.28) \quad \int_{\mathbb{R}} \mathcal{K}(x-y)|\chi_n|(y) \, dy \leq C_1 \delta^d + \delta, \quad \forall n \geq n_0 \quad \text{and} \quad |x| \geq R.$$

The same approach can be used to prove that for each  $n \in \{1, \dots, n_0 - 1\}$ , there is  $R_n > 0$  such that

$$(3.29) \quad \int_{\mathbb{R}} \mathcal{K}(x-y)|\chi_n|(y) \, dy \leq C_1 \delta^d + \delta, \quad |x| \geq R_n.$$

Hence, increasing  $R$ , if necessary, we must have

$$\int_{\mathbb{R}} \mathcal{K}(x-y)|\chi_n|(y) \, dy \leq C_1 \delta^d + \delta, \quad \text{for} \quad |x| \geq R, \quad \text{uniformly in} \quad n \in \mathbb{N}.$$

Since  $\delta$  is arbitrary, the proof is finished. ■

**Corollary 3.11.** *There is  $n_0 \in \mathbb{N}$  such that*

$$v_n(x, 0) < a, \quad \forall n \geq n_0 \quad \text{and} \quad x \in \Lambda_{\varepsilon_n}^c.$$

Hence,  $u_n(x) = v_n(x, 0)$  is a solution of  $(P'_{\varepsilon_n})$  for  $n \geq n_0$ .

**Proof.** By Lemma 3.8, we know that  $\varepsilon_n y_n \rightarrow x_0$ , for some  $x_0 \in \Lambda$ . Thereby, there is  $r > 0$  such that some subsequence, still denoted by itself,

$$(r - \varepsilon_n y_n, r + \varepsilon_n y_n) \subset \Lambda, \quad \forall n \in \mathbb{N}.$$

Hence,

$$(y_n - r/\varepsilon_n, y_n + r/\varepsilon_n) \subset \Lambda_{\varepsilon_n}, \quad \forall n \in \mathbb{N},$$

or equivalently

$$\Lambda_{\varepsilon_n}^c \subset (-\infty, y_n - r/\varepsilon_n) \cup (y_n + r/\varepsilon_n, +\infty), \quad \forall n \in \mathbb{N}.$$

Now, by Lemma 3.10, there is  $R > 0$  such that

$$w_n(x) < a, \quad \text{for} \quad |x| \geq R \quad \text{and} \quad \forall n \in \mathbb{N},$$

from where it follows,

$$v_n(x, 0) = \psi_n(x - y_n, 0) = w_n(x - y_n) < a, \quad \text{for} \quad x \in (-\infty, y_n - R) \cup (y_n + R, +\infty)$$

and  $\forall n \in \mathbb{N}$ .

On the other hand, we have that

$$\Lambda_{\varepsilon_n}^c \subset (-\infty, y_n - r/\varepsilon_n) \cup (y_n + r/\varepsilon_n, +\infty), \quad \forall n \in \mathbb{N}.$$

Thus, there is  $n_0 \in \mathbb{N}$ , such that

$$(-\infty, y_n - r/\varepsilon_n) \cup (y_n + r/\varepsilon_n, +\infty) \subset (-\infty, y_n - R) \cup (y_n + R, +\infty), \quad \forall n \geq n_0,$$

implying that

$$v_n(x, 0) < a, \quad \forall x \in \Lambda_{\varepsilon_n}^c \quad \text{and} \quad n \geq n_0,$$

finishing the proof. ■

## 4. PROOF OF THEOREM 1.1

By Theorem 3.6, we know that problem (AP) has a nonnegative solution  $v_\varepsilon$  for all  $\varepsilon > 0$ . Applying Corollary 3.11, there is  $\varepsilon_0$  such that

$$v_\varepsilon(x, 0) < a, \quad \forall x \in \Lambda_\varepsilon^c \quad \text{and} \quad \forall \varepsilon \in (0, \varepsilon_0),$$

that is,  $v_\varepsilon(\cdot, 0)$  is a solution of  $(P'_\varepsilon)$  for  $\varepsilon \in (0, \varepsilon_0)$ . Considering

$$u_\varepsilon(x) = v_\varepsilon(x/\varepsilon, 0), \quad \text{for} \quad \forall \varepsilon \in (0, \varepsilon_0),$$

is a solution for original problem  $(P_\varepsilon)$ .

If  $x_\varepsilon$  denotes a global maximum point of  $u_\varepsilon$ , it is easy to see that there is  $\tau_0 > 0$  such that

$$u_\varepsilon(x_\varepsilon) \geq \tau_0, \quad \forall \varepsilon > 0.$$

In what follows, setting  $z_\varepsilon = (x_\varepsilon - \varepsilon y_\varepsilon)\varepsilon^{-1}$ , we have that  $z_\varepsilon$  is a global maximum point of  $w_\varepsilon$  and

$$w_\varepsilon(z_\varepsilon) \geq \tau_0, \quad \forall \varepsilon > 0.$$

Now, we claim that

$$(4.1) \quad \lim_{\varepsilon \rightarrow 0} V(x_\varepsilon) = V_0.$$

Indeed, by Lemma 3.10, we know that

$$w_{\varepsilon_n}(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty \quad \text{uniformly in} \quad n \in \mathbb{N}.$$

Therefore,  $(z_\varepsilon)$  is a bounded sequence. Moreover, for some subsequence, we also know that there is  $x_0 \in \Lambda$  satisfying  $V(x_0) = V_0$  and

$$\varepsilon_n y_{\varepsilon_n} \rightarrow x_0, \quad \text{as} \quad n \rightarrow \infty.$$

Hence,

$$x_{\varepsilon_n} = \varepsilon_n z_{\varepsilon_n} + \varepsilon_n y_{\varepsilon_n} \rightarrow x_0, \quad \text{as} \quad n \rightarrow \infty,$$

implying that

$$V(x_{\varepsilon_n}) \rightarrow V_0, \quad \text{as} \quad n \rightarrow \infty,$$

showing that (4.1) holds.

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